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Resolution method for mixed integer bi-level linear problems based on decomposition technique

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Abstract In this article, we propose a new algorithm for the resolution of mixed integer bi-level linear problem (MIBLP). The algorithm is based on the decomposition of the initial problem into the restricted master problem (RMP) and a series of problems named slave problems (SP). The proposed approach is based on Benders decomposition method where in each iteration a set of variables are fixed which are controlled by the upper level optimization problem. The RMP is a relaxation of the MIBLP and the SP represents a restriction of the MIBLP. The RMP interacts in each iteration with the current SP by the addition of cuts produced using Lagrangian information from the current SP. The lower and upper bound provided from the RMP and SP are updated in each iteration. The algorithm converges when the difference between the upper and lower bound is within a small difference ε. In the case of MIBLP Karush–Kuhn–Tucker (KKT) optimality conditions could not be used directly to the inner problem in order to transform the bi-level problem into a single level problem. The proposed decomposition technique, however, allows the use of KKT conditions and transforms the MIBLP into two single level problems. The algorithm, which is a new method for the resolution of MIBLP, is illustrated through a modified numerical example from the literature. Additional examples from the literature are presented to highlight the algorithm convergence properties.

Keywords Bi-level optimization · Mixed integer linear programming · Benders decomposition · Active constraints

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1 Introduction

Hierarchical optimization deals with mathematical programming problems whose feasible set is implicitly determined by a sequence of nested optimization problems [26]. The most studied case is the bi-level programming problems and especially the linear case. In bi-level optimization we have two optimization levels the upper which is the leader and the lower optimization level which is the follower. The feasible region of upper optimization problem is determined by its own constraints plus the inner optimization problem. This problem in general is a nonconvex problem and its resolution is complicated. In bi-level optimization the leader controls a sub-set of decision variables (x) and the follower the other sub-set (y). If the leader chooses $x = x'$, then the follower responds with $y = y'$ as is shown in Fig. 1. For a given x, the follower solves the inner problem optimizing F_2. The leader examines the reactions of the follower for each feasible choice of x (the dashed line in x-axis). The set of all feasible solutions for the bi-level problem is the grey line in Fig. 1 and is called inducible region. The optimal solution in the linear case is an extreme point of the inducible region which is a nonconvex region [7]. The optimal solution of bi-level problem is a point that belongs to the inducible region where the upper level objective function F_1 takes its optimal value. In general a bi-level model can have continuous and integer decision variables. In order to simplify the presentation of the main characteristic of the bi-level problem we consider only continues variables: x which is a n_1—dimensional vector and y which is a n_2—dimensional vector and only constraints in the inner problem. This leads to the following bi-level linear problem:

$$
\begin{aligned}
& F: X \times Y \to R^1, \quad f: X \times Y \to R^1 \\
& \operatorname*{Max}_{x \in X} F_1(x, y) = c_1 x + d_1 y \\
& \text{st.} \operatorname*{Max}_{y \in Y} F_2(x, y) = c_2 x + d_2 y \\
& \quad \text{st. } g(x, y) = Ax + By \le b_1, \\
& \quad\quad \overline{y} \in Y = \{y : Cy \le b_2\} \\
& \quad\quad\quad \text{where} \quad c_1, c_2 \in \Re^{n_1}, d_1, d_2 \in \Re^{n_2}, b_1 \in \Re^{p}, b_2 \in \Re^{q}, \\
& \quad\quad\quad A \in \Re^{p \times n_1}, B \in \Re^{p \times n_2}, C \in \Re^{q \times n_2}
\end{aligned}
$$

From the leaders perspective this model can be viewed as a mathematical program with an implicit defined nonconvex constraint region given by the follower's sub-problem. In general in bi-level linear optimization problems the following regions and sets (cf. Fig. 1) are defined:

- The bi-level linear problem (BLP) constraints region is defied by the following region: $\Omega = \{(x, y) : x \in X, y \in Y, g(x, y) \le b_1\}$
- The projection of Ω onto the leader's decision space is $\Omega(X) = \{x \in |X : \exists y \ (x, y) \in \Omega\}$
- The follower's feasible region for $x \in X$fixed is $\Omega(x) = \{y : y \in Y, g(x, y) \le b_1\}$
- The follower's rational reaction set is $M(x) = \{y : \arg\max(f(\overline{y}) : \overline{y} \in \Omega(x))\}$
- The inducible region (IR) which correspond to the solution space of the bi-level problem is $\mathrm{IR} = \{(x, y) : x \in \Omega(X), y \in M(x)\}$

In order to ensure that the above bi-level problem is well posed we make the additional assumption that Ω is nonempty and compact and that for each decision taken by the leader, the follower can respond $(\Omega(x) \ne 0)$ [20]. The rational reaction set, $M(x)$, defines these responses while the IR, represents the set over which the leader may optimize. Thus, in terms of the above notation, the BLP can be written as: $\operatorname{Max}(F(x, y) : (x, y) \in \mathrm{IR})$. A bi-level

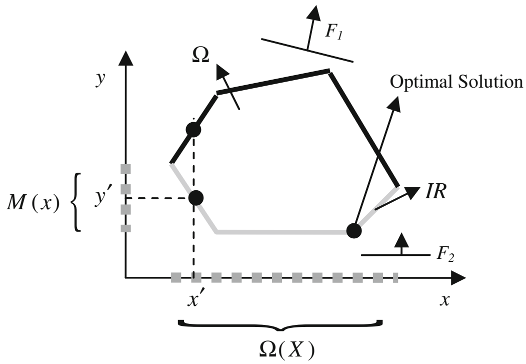

Fig. 1 Bi-level linear problem

feasible solution is a pair of $(\overline{x}, \overline{y})$ if $\overline{y} \in M(\overline{x})$ for the specific $\overline{x}$. An optimal solution for the bi-level problem is a point (x^*, y^*) if this point is a feasible point and for all bi-level feasible pairs $(\overline{x}, \overline{y}) \in IR$, $F(x^*, y^*) \geq F(\overline{x}, \overline{y})$.

If the problem involves integer and continuous variables and if all the variables (integer and continuous) are separable and appear in linear relations, then the bi-level problem corresponds to a mixed integer bi-level linear problem and in the general form can be described as follows:

$$\text{For} \quad \begin{array}{l} x \in X \subset R^n, \;\; y \in Y \subset R^m, \;\; \mathrm{z} \in Z_z\{0,1\}, \;\; \mathrm{w} \in Z_w\{0,1\}, \\ F_1\colon X \times Y \times Z_z \times Z_w \to R^1, \quad F_2\colon X \times Y \times Z_z \times Z_w \to R^1 \end{array}$$

$$\min_{x \in X, z \in Z_z} F_1(x, y) = c_1 x + d_1 y + r_1 \mathrm{z} + g_1 w$$

$$\text{st. } A_1 x + B_1 y + C_{1,} \mathrm{z} + Q_1 w \leq b_1,$$

$$\min_{y \in Y, w \in Z_w} F_2(x, y) = c_2 x + d_2 y + r_2 \mathrm{z} + g_2 w$$

$$\text{st. } A_2 x + B_2 y + C_2 \mathrm{z} + Q_2 w \leq b_2,$$

$$\text{where} \quad c_1, c_2 \in \Re^n, d_1, d_2 \in \Re^m, r_1, r_2 \in \Re^l, g_1, g_2 \in \Re^s, b_1 \in \Re^p, b_2 \in \Re^q,$$
$$A_1 \in \Re^{p \times n}, B_1 \in \Re^{p \times m}, C_1 \in \Re^{p \times l}, Q_1 \in \Re^{p \times s}, A_2 \in \Re^{q \times n}, B_2 \in R^{q \times m},$$
$$C_2 \in \Re^{q \times l}, Q_2 \in \Re^{q \times s}$$

In the literature, the exact methods developed for the solution of the mixed integer bi-level linear problem (MIBLP) have so far addressed a very restricted class of problems. There has been more attention in bi-level linear problems (BLP) where there no integer decision variables are involved in the inner problem. In general, we can partition the exact resolution techniques, for bi-level problems, in two groups, the enumeration and the reformulation techniques. The enumeration techniques are based on the property of bi-level problem, that the global optimum lies at a corner of the region that corresponds to feasible space defined by the upper and lower level constraints. The reformulation techniques transform the bi-level problem into a single level using for example the Karush–Kuhn–Tucker (KKT) optimality conditions of the lower level and introducing them as constraints in the upper level problem.

Extreme point algorithms are the basis for enumeration techniques applied to BLP. Every BLP with a finite optimal solution shares the important property that at least one optimal solution is attainted at an extreme point of the constraints region. This result was first established by Candler and Townsley [7] for the linear bi-level problem. The authors presented the “Kth-best” algorithm where the extreme points are evaluated in order to find the global optimal solution. The same idea is presented also by Bilias and Karwan [5]. Bard [1,2] and Bilias and Karwan [5] proved these results under the assumption that the constraints region is bounded. Savard [23] proved the same result in the case where also the upper level has constraints. Vincente et al. [28] have studied the induced regions of the quadratic bi-level problem and introduced the concepts of extreme induced region points and extreme induced region directions. Another algorithm of this group is the complementary pivot algorithm proposed by Bilias et al. [6]. Unfortunately this algorithm cannot, as suggested in [5], compute global solutions of BLP. Others extreme point algorithms are proposed by Chen and Florian [8], Papavassilopoulos [21] and Tuy et al. [27] that present algorithms based on global optimization techniques.

Reformulation techniques are developed from the need to solve the nonlinear or quadratic bi-level problems. The most commonly used reformulation technique is to replace the inner problem of the bi-level problem with KKT optimality conditions and append the resultant system to the upper level problem. In the case of BLP the problem is reformulated into a corresponding single level nonlinear programming. Shi et al. [25], Bialas and Karwan [5] and Hansen et al. [16] use the KKT optimality conditions in order to replace the inner problem and they propose different forms of branch and bound techniques for the resolution of the reformulated problem. Bard and Moore [3] proposed also a new branch and bound algorithm for the resolution of the linear quadratic bi-level problem. KKT reformulation approach has been proven to be a valuable analysis tool for bi-level problems. However there exists a serious deficiency for this approach when the upper level problem involves constraints in an arbitrary linear form. Shi et al. [24] shows this deficiency through some examples and proposed an extended KKT approach which resolves this problem. Another reformulation technique is the penalty function method [18] which is used to solve non-linear bi-level problems or to solve the problem of non-linearity using KKT condition. The complementary and slackness condition of the lower level problem is appended to the upper level objective with a penalty. Recently a new reformulation technique is developed using parametric programming [10]. In this approach parametric programming theory is used in order to transform the bi-level problem into a number of quadratic or linear problems. Finally Floudas et al. [12] presented a deterministic global optimization framework based on the ideas of feasible region convexification and branch and bound to address the problem of process feasibility and flexibility that correspond to bi-level nonlinear problems. Visweswaran et al. [29] proposed another global optimization approach for convex bi-level problems using decomposition based algorithm.

In many real systems, the leader may have to make discrete decisions. This type of decision can be described by the introduction of binary variables in the model. However, there has been very little attention in the literature on both the solution and the application of bi-level problems involving discrete decisions. In the literature, exact before the word methods developed for the solution of the mixed-integer BLP (MIBLP) have so far addressed a very restricted class of problems. Moore and Bard [20] developed a basic implicit enumeration scheme that finds good feasible solutions using relatively few iterations. The algorithm addresses mixed integer BLPP and uses a depth-first branch-and-bound approach incorporating some modifications in the typical depth-first branch-and-bound scheme used to solve mixed integer linear problems. In particular special fathoming rules were introduced to ensure the generation of valid upper bounds. The algorithm is very efficient for relatively small-scale problems; whereas for large scale problems a series of heuristics are proposed in an effort to strike a balance between accuracy and speed. For the solution of the mixed-integer BLP, another branch-and bound technique is developed by Wen and Yang [31], where only the outer problem has discrete decisions and the inner problem has continuous decisions.

Cutting plane and parametric solution approaches have been developed by Dempe [9] to solve problems where the inner level has a separable outer variable in its objective function only. Another interesting approach is presented by Faisca et al. [10]. The proposed algorithm is based on parametric programming theory and uses the basic sensitivity theorem. The main idea is to divide the follower's feasible area into different rational reaction sets, and search for the global optimum of a multi-parametric mixed integer linear programming problem in each area. Gümus and Floudas [15] introduced two deterministic global optimization methods that solve mixed integer nonlinear bi-level problems. The first is a global optimization approach to solve problems in which the outer level may be mixed-integer nonlinear and the inner level continuous nonlinear. The second is a global optimization approach to solve problems in which the outer level may involve general mixed-integer nonlinear functions. The inner level may involve functions that are mixed-integer nonlinear in outer variables and linear polynomial or multi-linear in inner integer variables and linear in inner continuous variables. The technique is based on the reformulation of the mixed integer inner problem as continuous via its convex hull representation and solving the resulting nonlinear bi-level optimization problem by a novel deterministic global optimization framework.

In this paper a new algorithm is presented for the resolution of the mixed integer linear bi-level problem. The algorithm is based on Benders decomposition technique which makes the use of KKT optimality conditions a valid reformulation procedure. Decomposition techniques are based on the idea of exploiting the decomposable structure of the problems in order to facilitate the solution of the initial problem through a series of smaller sub-problems. In this case one of the sub-problems is the slave problem (SP) which is obtained by fixing a number of decision variables of the initial problem (MIBLP) to a feasible value and the second one is the restricted master problem (RMP) which gives the optimal solution after the addition of cuts. The only assumption of the proposed algorithm is that although integer variables could appear in both levels of MIBLP, they should be controlled by the upper optimization problem. Then in each iteration of the algorithm, the SP gives a new valid cut to the RMP which converges to the optimal solution. If the RMP optimality condition is not satisfied by the bounded solution obtained by SP, the RMP sends the updated information to the SP which produces another cut for RMP and the algorithm continues until the RMP optimality condition is satisfied [4,19]. In the proposed algorithm, the same idea is used in order to transform the initial mixed integer bi-level linear problem into two models where the first one is a mixed integer linear problem (RMP) and the second one a bi-level linear problem (SP) where the binary variables are fixed. The KKT optimality conditions are not applicable when there are integer variables in the inner problem of the bi-level problem. That means that the proposed decomposition of the initial problem into RMP and SP permits the use of KKT conditions because the resulted SP problem does not have any integer variables in the inner level. The use of KKT conditions for the mixed integer bi-level problem using Benders decomposition makes this algorithm novel. It should be noticed that the proposed algorithm can be also used for the cases of linear bi-level problems where KKT optimality conditions can be directly applied. The main advantage of the proposed approach however is for the resolution of large scale problems where decomposition methods are more beneficial.

The rest of the paper is organized as follows. In Sect. 2, we present the proposed algorithm followed by a numerical example in Sect. 3 to illustrate and further clarify the basic steps of the approach. In Sect. 4, we present the basic conclusions and give our perspective for the suitability of the presented work. Finally, in the appendix the theoretical background of the proposed algorithm is given.

2 Algorithm for mixed integer bi-level linear problem

2.1 Motivation for the development of the proposed algorithm

Our motivation for the resolution of this class of problems is the cross validation of experimental data for the development of the regulatory networks. Foteinou et al. [13] propose a systematic construction of alternative regulatory architectures and a consistency metric for assessing the robustness and specific transcription factors. The authors evaluate the biological implications of the multiple alternative structures in their biological context and demonstrate how a systematic framework can define the basis for a consistent hypothesis generation mechanism related to putative regulatory interactions. They use cross-validation of the experimental data in order to minimize the error in the construction of the gene network. Cross validation is the statistical practice of partitioning a sample of data into subsets such that the analysis is initially performed on a single subset, while the other subset is retained for subsequent use in confirming and validating the initial analysis. The cross validation of the experimental data for the development of the regulatory networks is done in two steps. In the first one, we determine the structure of the genes network using the first sub-set of data and in the second one, we confirm the obtained structure. Our goal is to perform the cross validation in only one step, performing the development and the validation of the genes network at the same time. To achieve this goal a bi-level optimization is proposed where the leader does the validation of the developed gene network and the follower determines the structure of the gene network.

2.2 Basic idea of the algorithm

The basic idea of the proposed algorithm is to decompose the initial problem to the following sub-problems, a series of problems named slave problems (SP) and the restricted master problem (RMP) which converge to the optimal solution. The proposed algorithm is based on Benders decomposition method. The algorithm produces a cut, in each iteration, which is added to the RMP. The RMP is a relaxation of our initial problem and gives a lower bound (LB) for the algorithm when the initial problem is a minimization problem. The algorithm uses the KKT optimality conditions in order to transform the initial restricted (by fixing the value of the integer variables) bi-level problem to a single level problem. The restriction of the initial problem (SP) gives an upper bound (UB) if the initial MIBLP is a minimization problem. These characteristics show that our algorithm belongs to the class of algorithms known as reformulation algorithms.

The first step of the algorithm is to decompose the initial mixed integer bi-level linear problem to the RMP and SP. The algorithm starts by fixing the integer variables to specific values ($z = \overline{z}$) and constructing the current slave problem (SP($\overline{z}$)) which is a bi-level linear problem. This problem is then reformulated to a mixed integer linear problem using the KKT optimality conditions of the inner problem and the active set strategy approach [14]. The solution of this problem provides information about the active constraints which are used to build the corresponding linear problem (LP($\overline{z}$)). This linear problem, with the same active constraints derived by the current slave problem (SP($\overline{z}$)), gives an upper bound (in the case of minimization) to the solution of the original problem. Using the optimal dual values of the current linear problem (LP($\overline{z}$)) we construct a cut which is added to the RMP. In general one of the three following cuts could be generated (cf. Appendix A):

- Optimality cut: when the current LP(z) gives a feasible solution;
- Feasibility cut: when the current LP(z) gives an infeasible solution;
- Exclusion Cut: when the current LP(z) gives a feasible solution but the cut does not restrict the RMP.

After the addition of the current cut, the augmented RMP is solved to obtain a new lower bound (in the case of minimization). In the next step the RMP optimality condition ($\text{UB} - \text{LB} < \varepsilon$) is checked. If it is satisfied the algorithm stops, otherwise the algorithm continues using the current solution obtained by the RMP in order to construct a new SP. The algorithm continues until the RMP optimality condition is satisfied. The flowchart of the proposed algorithm is presented in Fig. 2.

2.3 Validity of the upper and lower bounds

A relaxation of the initial bi-level problem is formulated by eliminating a set of constraints. In the proposed approach the RMP involves the set of constraints that have only integer variables. The constraints involving continuous and integer variables, and the inner objective function are eliminated. Thus the resulted RMP provides in each iteration a valid lower bound (LB) in the case of minimization.

A restriction of the original problem is then needed to provide a valid upper bound (for the case of minimization) in each iteration of the proposed algorithm. In general for a single level linear program a restriction can be obtained by fixing one or more of the decision variables or in general by adding inequalities to restrict the value of one or more of the decision variables. In the case of bi-level optimization programming where the problem is nonconvex in order to generate a restriction of the initial bi-level problem, the new bi-level problem should have a New IR (NIR) which is a restriction of the original IR. As presented above, the IR of the bi-level problem is defined by: $x \in \Omega(x)$ and $y \in M(x)$ which means that the IR is defined by $\Omega(x)$ and the reaction of the follower for each $y \in \Omega(x)$. Restricting a variable decision updated regions $\Omega'(x)$ and $M'(x)$ are obtained. In order to make sure that the resulted bi-level problem is a restriction of the original problem, $\Omega'(x)$ and $M'(x)$ should satisfy the following conditions: $\Omega'(x) \subseteq \Omega(x)$ and $M'(x) \subseteq M(x)$.

In bi-level optimization the leader examines the reactions of the follower for each feasible choice of its variables. Thus, restricting a variable which is controlled by the leader results in a restriction of the constraints region (Ω). Consequently in the new bi-level problem the leader is looking for the reaction of the follower in a region which is a restriction of $\Omega(x)$ or otherwise the resulted projection of the new constraint region on the leader's decision space is a restriction of the initial region ($\Omega'(x) \subseteq \Omega(x)$). A direct result of this observation is that the follower's rational reaction set is a restriction of the initial one ($M'(x) \subseteq M(x)$) and thus the NIR is a sub-set of the initial IR. Thus the SP provides a valid cut and a valid upper bound.

Restricting a variable which is controlled by the follower even if this results in a restriction of the constraints region (Ω), and $\Omega(x)$ remains the same, the resulted SP is not guarantee to provide a valid upper bound. The reason is that fixing a variable within the constraints region of the MIBLP does not guarantee that the selected value belongs to $M(x)$. In order to obtain a restriction of the initial problem by fixing a decision variable controlled by the follower a preliminary analysis should be performed before each iteration of the proposed algorithm to ensure that the resulted reaction set of the follower is a sub-set of the initial set ($M'(x) \subseteq M(x)$). This extension will be the subject of future publication and is not addressed in this paper.

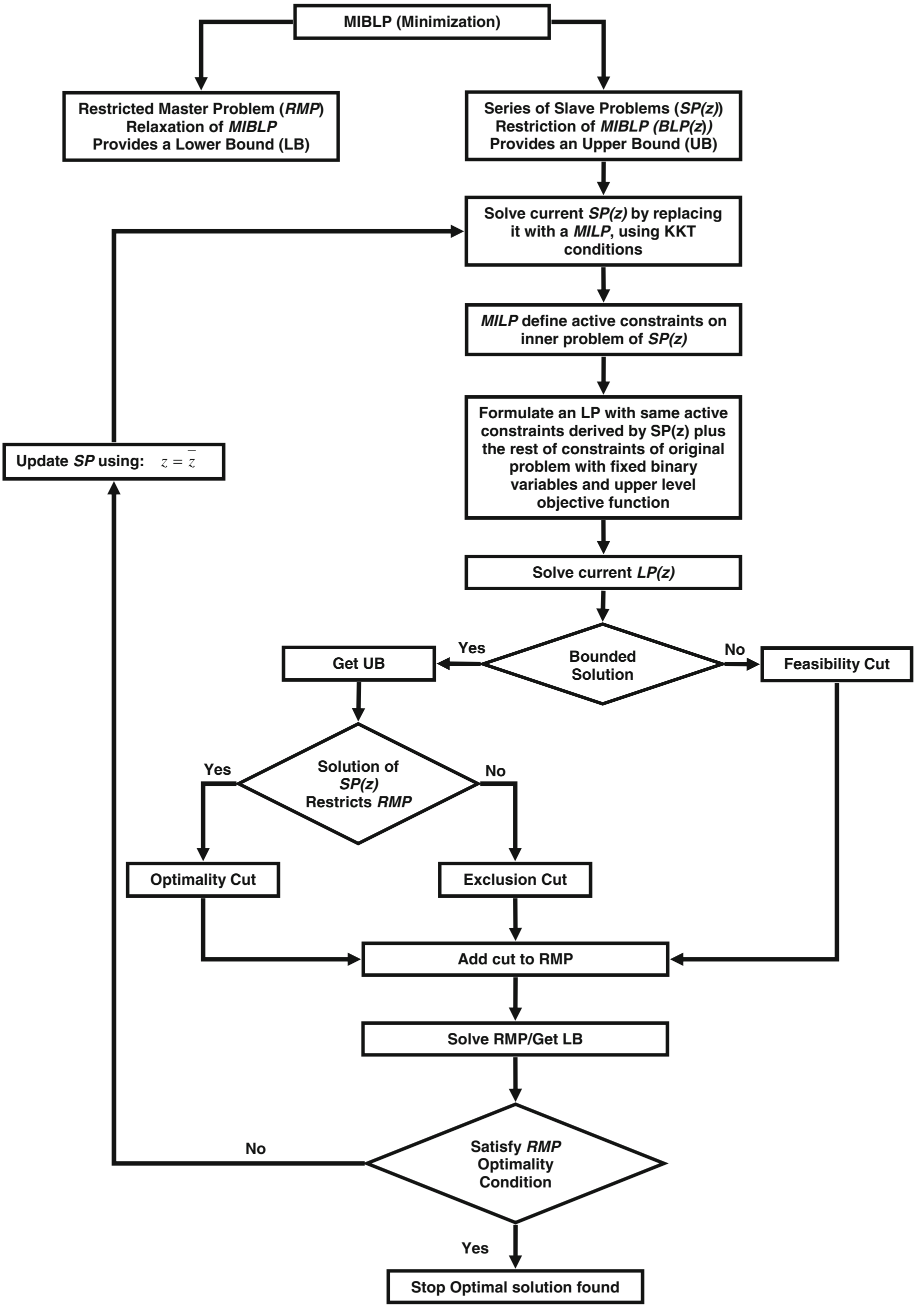

Fig. 2 Flowchart of the proposed algorithm

3 Numerical examples

3.1 Illustrative example

In this section the algorithm is illustrated using an example presented in [31]. The example is modified to reduce the number of variables so that it can be represented in two dimensions for illustrative purposes. After fixing some variables to the optimal values, the model takes the following form:

$$\begin{aligned}
&\min_{x_2, y_2, y_3} F_1(x_2, y_2, y_3) = -60x_2 - 10y_2 - 7y_3 \\
&\text{st.} \quad \min_{y_2, y_3} F_2(y_2, y_3) = -60y_2 - 8y_3 \\
&\text{st.} \quad g_1\colon 10x_2 + 2y_2 + 3y_3 \le 225, \\
&\qquad g_2\colon 5x_2 + 3y_2 \le 230, \\
&\qquad\quad g_3\colon 5x_2 + y_3 \le 85. \\
&x_2 = \{0, 1\} / y_2, y_3 \ge 0
\end{aligned}$$

This bi-level problem is decomposed into the following master and slaves problems where the binary variable x_2 is fixed ($x_2 = \overline{x_2}$) (cf. Appendix A, problem P_2) and:

Restricted Master Problem (RMP):

$$\begin{aligned}
&\text{Min } F_3(x_2, \xi) = \xi - 60x_2 \\
&\text{st.} \quad -M \le \xi \le M \\
&x_2 = \{0, 1\} \\
&M = 1200
\end{aligned}$$

Slave Problem SP($\overline{x_2}$):

$$\begin{aligned}
&\text{Min } F_1(y_2, y_3) = -10y_2 - 7y_3 - 60\overline{x_2} \\
&\quad \text{st.} \quad \text{Min } F_2(x, y) = -60y_2 - 8y_3 \\
&\qquad \text{st.} \quad 2y_2 + 3y_3 \le 225 - 10\overline{x_2}, \\
&\qquad\qquad 3y_2 \le 230 - 5\overline{x_2}, \\
&\qquad\qquad y_3 \le 85 - 5\overline{x_2}. \\
&\qquad\qquad y_2, y_3 \ge 0
\end{aligned}$$

In the first iteration of the algorithm, we fix arbitrarily the binary variable $x_2 = 0$ and this results in the following problem SP(0):

$$\begin{aligned}
&\text{Min } F_1(y_2, y_3) = -10y_2 - 7y_3 \\
&\text{st.} \quad \text{Min } F_2(y_2, y_3) = -60y_2 - 8y_3 \\
&\text{st.} \quad 2y_2 + 3y_3 \le 225, \\
&\qquad 3y_2 \le 230, \\
&\qquad y_3 \le 85. \\
&\qquad y_2, y_3 \ge 0
\end{aligned}$$

The solution space of the initial RMP (the two dashed double-lines) and the current constraint region of the SP (the grey space) with $x_2 = 0$ (SP(0)) are depicted in Figs. 3 and 4, respectively. Notice that the constraint space of the current SP is not the feasible set (the inducible region) of BLP. The inducible region of SP(0) is the dashed line and one of the extreme point of the inducible region is the optimal solution of SP(0).

The initial value of the lower and upper bound of the algorithm are fixed to LB $= -1260$ and UB $= \infty$. In order to solve the SP(0) a reformulation technique is needed. The Karush–

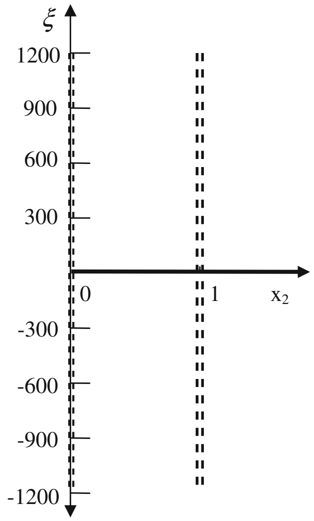

Fig. 3 RMP solution space

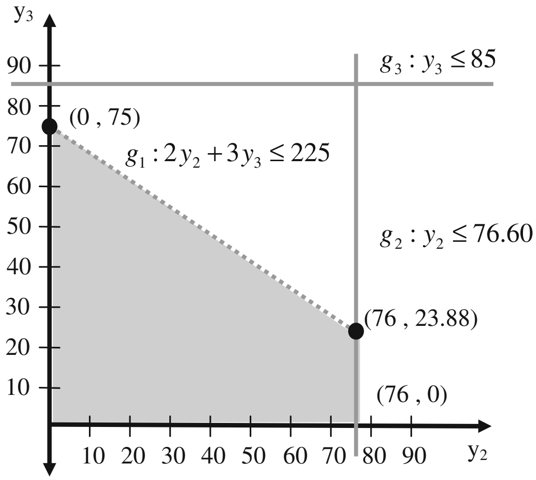

Fig. 4 SP(0) solution space

Kuhn–Tucker method [24] and active set strategy [14] are used and the BLP is transformed into the following MILP (cf. Appendix A, problem P_3):

$$
\begin{aligned}
&\text{Min } F_1(y_2, y_3) = -10y_2 - 7y_3\\
&\text{st.} \quad g_1: 2y_2 + 3y_3 \le 225,\\
&\qquad g_2: 3y_2 \le 230,\\
&\qquad g_3: y_3 \le 85,
\end{aligned}
$$

$$
\text{Stationary conditions} \quad \begin{cases} 2u_1 + 3u_2 - u_4 = 60, \\ 3u_1 + u_3 - u_5 = 8, \end{cases}
$$

$$\text{Complimentarity conditions}\quad \begin{cases} u_1 - Mv_1 \le 0, 225 - 2y_2 - 3y_3 - M(1 - v_1), \\ u_2 - Mv_2 \le 0, 230 - 3y_2 - M(1 - v_2), \\ u_3 - Mv_3 \le 0, 85 - y_3 - M(1 - v_3), \\ u_4 - Mv_4 \le 0, y_2 - M(1 - v_4), \\ u_5 - Mv_5 \le 0, y_3 - M(1 - v_5). \end{cases}$$
$$y_2, y_3 \ge 0, u_1, u_2, u_3, u_4, u_5 \ge 0, v_1, v_2, v_3, v_4, v_5 = \{0, 1\}.$$

The resolution of this problem shows that the inducible region of the bi-level problem is the line: $2y_2 + 3y_3 = 225$: (c.f. Fig. 4) That means that the first constraint of the bi-level problem is an active constraint. The corresponded linear problem takes the following form (cf. Appendix A, problems P_4, P_5):

$$\begin{aligned} &\text{Min } F_1(y_2, y_3) = -10y_2 - 7y_3 \\ &\text{st.}\quad g_1\text{: } 2y_2 + 3y_3 = 225, \leftarrow w_1 \\ &\qquad g_2\text{: } 3y_2 \le 230, \leftarrow w_2 \\ &\qquad g_3\text{: } y_3 \le 85, \leftarrow w_3 \\ &\qquad y_2, y_3 \ge 0 \end{aligned}$$

The optimal solution of this problem gives a new upper bound $\text{UB}_{new} = -933.16$ as proved in Sect. 2.3. The optimality condition ($\text{LB} = -1260 \neq -933, 16 = \text{UB}$) is not satisfied and the current corresponded LP produce the first cut (cf. Appendix A, case 3). The value of dual variables (w_1, w_2, w_3) produce the following cut which is added to the RMP producing the following augmented RMP:

$$\begin{aligned} &\text{Restricted Master Problem:} \\ &\text{Min } F_3(x_2, \xi) = \xi - 60x_2 \\ &\text{st.}\quad -M \le \xi \le M \\ &(225 - 10x_2)w_1 + (230 - 5x_2)w_2 + 85w_3 - \xi \le 0 \\ &\text{where}\quad w_1 = -2.3334, w_2 = -1, 7778, w_3 = 0 \\ &x_2 = \{0, 1\} \end{aligned}$$

The new RMP with this cut (dashed line) has the solution space as presented in Fig. 5.

The resolution of the RMP gives a new lower bound $\text{LB}_{\text{new}} = -961.69$ as shown in Sect. 2.3. Comparison of the UB and the new LB does not satisfy the RMP optimality condition of the algorithm since $\varepsilon = 0$. Thus the algorithm continues by using the solution provided by the resolution of the RMP for the integer variable decision ($x_2 = 1$). The new SP(1) takes the following form:

$$\begin{aligned} &\text{Slave Problem SP}(x_2 = 1): \\ &\text{Min } F_1(y_2, y_3) = -10y_2 - 7y_3 - 60 \\ &\text{st.}\quad \text{Min } F_2(y_2, y_3) = -60y_2 - 8y_3 \\ &\qquad \text{st.}\quad 2y_2 + 3y_3 \le 215, \\ &\qquad\qquad 3y_2 \le 225, \\ &\qquad\qquad y_3 \le 80. \\ &\qquad\quad y_2, y_3 \ge 0 \end{aligned}$$

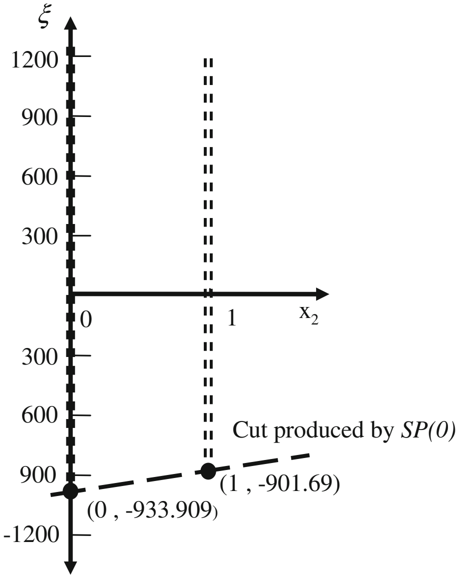

Fig. 5 RMP solution spaceat the second iteration

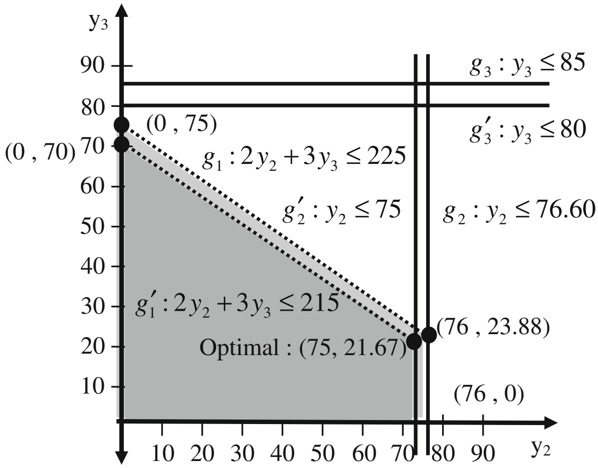

Fig. 6 SP(1) solution space

The resolution of this new SP(1) gives a new inducible region (the line: $2y_2 + 3y_3 = 215$, in Fig. 6) and a new upper bound: $\text{UB}_{\text{new}} = -961.69$ (cf. Appendix A, problems P_4, P_5). At this point the algorithm stops because the LB and the UB obtained by the RMP and the SP are equal. The optimal solution of the problem is the point $(y_2, y_3) = (75, 21.67)$, $x_2 = 1$.

3.2 Numerical examples

As noted in the introduction they are few papers in the literature that address the class of bi-level problems considered in this paper. Table 1 illustrates the comparison between the

Table 1 Results of numerical examples

	Example	Results in the reference	Results of the proposed approach	Proposed approach CPU (s)
1	U. P. Wen and Y. H. Yang [31]	$F_{\text{upper}} = -1011.69$, $f_{\text{lower}} = -4673.34$	$F_{\text{upper}} = -1011.69$, $f_{\text{lower}} = -4673.34$	2.42
2	J. F. Bard [1]	$F_{\text{upper}} = 41.2$, $f_{\text{lower}} = -9.2$	$F_{\text{upper}} = 41.2$, $f_{\text{lower}} = -9.2$	2.92
3	P. Hansen et al. [16]	$F_{\text{upper}} = 18.4$, $f_{\text{lower}} = -1.8$	$F_{\text{upper}} = 18.544$, $f_{\text{lower}} = -1.797$	2.56
4	A. Haurie et al. [17]	$F_{\text{upper}} = -27$, $f_{\text{lower}} = 3$	$F_{\text{upper}} = -27$, $f_{\text{lower}} = 3$	1.39

proposed procedure for solving the MIBLP and the already existing techniques. Note that the CPU times are not mentioned on the referenced papers for these examples. All results presented in this paper have been obtained on Pentium (R) 4, CPU 2.40 GHz, RAM 1 GB and CPLEX 10 using a C++ implementation of the proposal approach.

Based on the results obtained for these examples it should be noticed that first the algorithm is successful in obtaining the global solution for each problem requiring 2 iterations and only very small resolution time. Moreover, the approach obtains better solution in example 3 than the solution obtained in the reference paper. For the third example the optimal solution presented in the reference example is $F_{\text{upper}} = 18.4$ and $f_{\text{lower}} = -1.8$ with $(x_1, x_2, y_1, y_2, y_3) = (0.2, 0.8, 0, 0.2, 0.8)$ and the proposed algorithm gives a little better optimal solution: $F_{\text{upper}} = 18.544$ and $f_{\text{lower}} = -1.797$ with $(x_1, x_2, y_1, y_2, y_3) = (0.533, 0.8, 0, 0.197, 0.8)$. We should notice that the third example is a maximization problem and that the optimal solution obtained by the algorithm presented in this paper is the same as the solution presented in others papers that used the same numerical example to prove the efficiency of their proposed method [30].

4 Summary and future directions

In this paper, an exact algorithm for the resolution of the mixed integer bi-level linear problem (MIBLP) is presented. The algorithm is based on Benders decomposition method and it is a new alternative for the exact resolution of the mixed integer bi-level linear problem. The presented algorithm can be considered as a reformulation algorithm based on the use of KKT optimality conditions for the resolution of the involved bi-level linear problem after the decomposition. In the developed algorithm the initial MIBLP is decomposed into several sub models, the restricted master problem (RMP) and a series of bi-level linear problems named slave problems (SP). The RMP is a relaxation of the initial MIBLP and in the case of minimization provides a lower bound for the algorithm. The SP represents a restriction of the initial problem because they are derived from the initial MIBLP by fixing the integer variables controlled by the upper optimization problem. Thus, the solution of the SP in each iteration provides an upper bound in the case of minimization. The RMP results in the optimal solution after the addition of cuts produced form the SP. The convergence criterion is satisfied when the difference between the upper and lower bound of the algorithm is below a pre-specified tolerance.

The next step of this research project is to improve the performance of the algorithm so that it can be applied to large scale problems. In general in decomposition methods the convergence is related to the form of cuts produced in each iteration. An additional complication in this algorithm is that in each iteration the algorithm constructs and solves three different problems: the first one is for the resolution of the SP, the second one for the production of the cut and the update of the upper bound and the third is for the resolution of the RMP which provide also a lower bound. The RMP and the corresponding problem for the resolution of the bi-level linear SP are mixed integer linear problems, which is more complicated than the corresponding linear problems (cf Appendix A, problem P_4) used for the production of cuts. To improve the solution procedure we are investigating the possibility of solving these models in parallel. In particular we will consider the solution of multiple SP problems that correspond to different combinations of integer variables.

In addition we will examine the utilization of parallel optimization for the generation of more than one sufficient cut in each iteration of the algorithm as present in Saharidis et al. [22]. The multi generation of cuts decrease the number of algorithm's iterations decreasing in general the resolution time. It seems also interesting to study under which assumption the proposed algorithm can be applied for the case of mixed integer bi-level nonlinear problems. The proposed algorithm using linearization techniques as presented in [24] can provide good solutions for the case of nonlinear objective function or nonlinear constraints. Finally in our future research interest is to define a preliminary analysis for the case where the integer decision variables are controlled by the follower. This analysis will be performed before each iteration of the proposed algorithm in order to ensure that the resulted inducible region is a sub-set of the initial inducible region.

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Appendix A: Theoretical background of the proposed algorithm

Using the following notation the basic steps of the algorithm are described below:

Index

- $m = 1, \ldots, 4$ denotes the constraints in the illustrated example;
- $l = 1, \ldots, L$ denotes the number of extreme rays of the dual slave problem;
- $p = 1, \ldots, P$ denotes the number of extreme rays of the dual slave problem with $z = z_1$;
- $k = 1, \ldots, K$ denotes the number of extreme rays of the dual slave problem with $z = z_2$;
- $r = 1, \ldots, R$ denotes the number of extreme rays of the dual slave problem with $z = z_n$;
- $i = 1, \ldots, I$ denotes the number of extreme points of the dual slave problem with $z = z_1$;
- $j = 1, \ldots, J$ denotes the number of extreme points of the dual slave problem with $z = z_2$;
- $t = 1, \ldots, T$ denotes the number of extreme points of the dual slave problem with $z = z_n$.

Auxiliary decision variables

- $v_m = 1$ if the corresponding constraint m is active otherwise takes the value of zero;
- u_m: dual value of the m constraint;
- w_m: Lagrangian multiplier m constraint use for the active constraint strategy.

Parameter

- M: big value number.

$R^{f_1+f_2+f_3+\cdots} = R^{f_1} \cup R^{f_2} \cup R^{f_3} \cup R^{\cdots}$

$b_1 \in R^{b_1}, b_2 \in R^{b_2}, x \in X \subset R^x, y \in Y \subset R^y, s \in S \subset R^{b_1+b_2}, u \in U \subset R^{b_1+b_2}, v \in V \subset R^{b_1+b_2}$

$z \in Z\{0,1\} \subset R^z, bv \in Z\{0,1\} \subset R^v, h \in H \subset R^h$

$w_1 \in W_1 \subset R^{w_1}, w_2 \in W_2 \subset R^{w_2}, w_3 \in W_3 \subset R^{w_3}, w_4 \in W_4 \subset R^{w_4}$

$c_1 \in R^x, c_2 \in R^y, c_3 \in R^z$

$F_1, F_2 : X \times Y \times Z \to R^1$

$A_1 \in R^{b_1\times x}, B_1 \in R^{b_1\times y}, E_1 \in R^{b_1\times z}, Q_1 \in R^{b_1\times b_1}, C_1 = I \in R^{h\times h}$

$A_2 \in R^{b_2\times x}, B_2 \in R^{b_2\times y}, E_2 \in R^{b_2\times z}, Q_2 \in R^{b_2\times b_2}, C_2 = I \in R^{h\times h}$

$$c^T = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}, \ c^T \in R^c = R^{x+y+z}, D = \begin{bmatrix} A_1 & B_1 & Q_1 \\ A_2 & B_2 & Q_2 \end{bmatrix}, \ D \in R^D = R^{(b_1+b_2)\times(x+y+b_1+b_2)}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \ b \in R^b = R^{b_1+b_2}, g = \begin{bmatrix} x \\ y \\ s \end{bmatrix}, \ g \in R^g = R^{x+y+b_1+b_2}, E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \ E \in R^E = R^{b_1+b_2}$$

$$c'^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c'^T \in R^{c'} = R^{x+y+z+h}, \ D' = \begin{bmatrix} A_1 & B_1 & Q_1 & C_1 \\ A_2 & B_2 & Q_2 & C_2 \end{bmatrix}, D' \in R^{D'} = R^{(b_1+b_2)\times(x+y+h+b_1+b_2)}$$

$$g' = \begin{bmatrix} x \\ y \\ s \\ h \end{bmatrix}, \ g' \in R^{g'} = R^{x+y+h+b_1+b_2}$$

Let's consider the following model for the *MIBLP*:

$$P_1 : \begin{cases} \min\limits_{x,z} F_1(x) = c_1 x \\ \text{st.} \\ \min\limits_{y} F_2(x) = c_2 y \\ \quad \text{st.} \\ \qquad A_1 x + B_1 y + E_1 z \le b_1 \\ \qquad A_2 x + B_2 y + E_2 z \le b_2 \\ \qquad x, y \ge 0 \qquad z = \{0,1\} \end{cases}$$

Note that constraints in the upper level as well as binary variable in the objective function can be also considered. Fixing the values of the binary variables $z = \bar{z}$, we get the following bi-level linear problem (BLP):

$$
\mathrm{P}_2: \begin{cases} \underset{\mathrm{x}}{\mathrm{Min}}\ F_1(x) = c_1 x \\ \text{st.} \\ \qquad \underset{y}{\mathrm{Min}}\ F_2(x) = c_2 y \\ \qquad \text{st.} \\ \qquad\qquad A_1 x + B_1 y \le b_1 - E_1 \overline{z} \to w_1 \\ \qquad\qquad A_2 x + B_2 y \le b_2 - E_2 \overline{z} \to w_2 \\ \qquad\qquad -y \le 0 \to w_3 \\ \qquad\qquad -x \le 0 \to w_4 \end{cases}
$$

Using KKT conditions [25,5,16] and the active constraints strategy [14], we can transform this BLP to a mixed integer linear problem (MILP):

$$
P_3: \begin{cases} \mathrm{Min}\ F_1(x) = c_1 x \\ \text{st.} \\ \qquad A_1 x + B_1 y \le b_1 - E_1 \overline{z} \\ \qquad A_2 x + B_2 y \le b_2 - E_2 \overline{z} \\ \qquad w_1 - Mbv_1 \le 0, \quad w_2 - Mbv_2 \le 0 \\ \qquad w_3 - Mbv_3 \le 0, \quad w_4 - Mbv_4 \le 0 \\ \qquad (b_1 - E_1 \overline{z}) - A_1 x - B_1 y - M(1 - bv_1) \le 0 \\ \qquad (b_2 - E_2 \overline{z}) - A_2 x - B_2 y - M(1 - bv_2) \le 0 \\ \qquad y - M(1 - bv_3) \le 0, \quad x - M(1 - bv_4) \le 0 \\ \qquad w_1 A_1 + w_2 A_2 - w_4 = 0 \\ \qquad w_1 B_1 + w_2 B_2 - w_3 = -c_2 \\ \qquad x, y, w_1, w_2, w_3, w_4 \ge 0 \quad bv_1, bv_2, bv_3, bv_4 = \{0, 1\} \end{cases}
$$

From the solution of P_3, we find which constraints are active and we transform the initial problem P_1 to the following linear problem P_4 where we assume that the first group of constraints is active.

$$
P_4: \begin{cases} \mathrm{Min}\ F_1(x) = c_1 x \\ \text{st.} \\ A_1 x + B_1 y = b_1 - E_1 \overline{z} \\ A_2 x + B_2 y \le b_2 - E_2 \overline{z} \\ x, y \ge 0 \end{cases}
$$

The general form of P_4 can be reformulated as follows (P_5):

$$
P_5 \begin{cases} \mathrm{Min}\ F_1(x) = c_1 x \\ \text{st.} \\ A_1 x + B_1 y + Q_1 s = b_1 - E_1 \overline{z} \\ A_2 x + B_2 y + Q_2 s = b_2 - E_2 \overline{z} \\ x_1 y_1 s \ge 0 \end{cases}
$$

where Q_1, Q_2 are matrices where all the elements are equal to zero except the elements in the diagonal that correspond to non-active constraints and are equal to 1. In order to simplify the presentation we transform the problem using the notation presented in the nomenclature. Thus problem P_5 takes the follow form:

$$P_5' \begin{cases} \text{Min } F_1(x) = cg \\ \text{st.} \\ Dg = b - E\overline{z} \\ g \geq 0 \end{cases}$$

In the decomposition procedure we propose, we cannot choose the variable z arbitrarily. Problem P_2 (and P_3) should have at least a non empty solution set for z. To express this condition, we use P_3' which gives the best feasible solution minimizing the auxiliary variables h.

$$P_3' : \begin{cases} \text{Min } h \\ \text{st.} \\ \quad A_1x + B_1y - C_1h \leq b_1 - E_1\overline{z} \\ \quad A_2x + B_2y - C_2h \leq b_2 - E_2\overline{z} \\ \quad w_1 - Mbv_1 \leq 0, w_2 - Mbv_2 \leq 0 \\ \quad w_3 - Mbv_3 \leq 0, w_4 - Mbv_4 \leq 0 \\ \quad (b_1 - E_1\overline{z}) - A_1x - B_1y + C_1h - M(1 - bv_1) \leq 0 \\ \quad (b_2 - E_2\overline{z}) - A_2x - B_2y + C_2h - M(1 - bv_2) \leq 0 \\ \quad y - M(1 - bv_3) \leq 0, \quad x - M(1 - bv_4) \leq 0, \quad h - M(1 - bv_5) \leq 0 \\ \quad w_1A_1 + w_2A_2 - w_4 = 0 \\ \quad -w_1C_1 - w_2C_2 - w_5 = 0 \\ \quad w_1B_1 + w_2B_2 - w_3 = -c_2 \\ \quad x, y, w_1, w_2, w_3, w_4, w_5 \geq 0 \quad bv_1, bv_2, bv_3, bv_4, bv_5 = \{0, 1\} \end{cases}$$

From P_3', we find which constraints are active and we transform the initial problem P_1 to the following linear problem P_5'':

$$P_5'' : \begin{cases} \text{Min } c'g' \\ \text{st.} \\ D'g' = b - E\overline{z} \\ g' \geq 0 \end{cases}$$

The necessary and sufficient condition for P_5'' to have at least a non empty solution for z is given by lemma of *Farkas* and *Minkowski*. At each constraint i of P_5'' corresponds a dual variable u_i (not sing restricted), then the lemma of *Farkas* and *Minkowski* states:

The problem P_5'' has a solution $g' \geq 0$ if and only if $u(b - E\overline{z}) \leq 0$ for all u for which $uD' \leq 0$ holds.

We should notice that for each z the matrix D' is different based on the values of matrix Q. That means that the matrix D' is related to the value of z, so it can be stated as$D'(z)$. For each z the cone $U(z) = \{u/uD'(z) \leq 0\}$ has a finite number of generators which we denote as $u_1^z, \ldots, u_L^z$. The necessary and sufficient condition of the Farkas and Minkowski lemma for each $z_1, z_2, \ldots, z_n$ is then equivalent to the system of inequalities:

$$\begin{array}{l} \text{for } z_1 \begin{cases} u_1^{z_1} \cdot (b - E \cdot z_1) \le 0 \\ u_2^{z_1} \cdot (b - E \cdot z_1) \le 0 \\ \quad \cdots\cdots\cdots\cdot \\ u_P^{z_1} \cdot (b - E \cdot z_1) \le 0 \end{cases} \\ \text{for } z_2 \begin{cases} u_1^{z_2} \cdot (b - E \cdot z_2) \le 0 \\ u_2^{z_2} \cdot (b - E \cdot z_2) \le 0 \\ \quad \cdots\cdots\cdots\cdot \\ u_K^{z_2} \cdot (b - E \cdot z_2) \le 0 \end{cases} \\ \text{for } z_n \begin{cases} u_1^{z_n} \cdot (b - E \cdot z_n) \le 0 \\ u_2^{z_n} \cdot (b - E \cdot z_n) \le 0 \\ \quad \cdots\cdots\cdots\cdot \\ u_R^{z_n} \cdot (b - E \cdot z_n) \le 0 \end{cases} \end{array} \tag{I}$$

Suppose that P_3 has a solution for given $z = \overline{z}$, then the dual of P_5' reads:

$$\text{Dual of } P_5' \begin{cases} \text{Max } F_3(v) = v(b - E\overline{z}) \\ \text{st.} \\ vD(\overline{z}) \le c \\ v \text{ of any sign} \end{cases}$$

where v is the vector of dual variables associated with the constraints of P_5'; $(v_1^{z_1}, \ldots, v_P^{z_1})$, $(v_1^{z_2}, \ldots, v_K^{z_2}), \ldots, (v_1^{z_{n1}}, \ldots, v_R^{z_n})$ are the extreme rays of the cones $vD(z_1) \le 0$, $vD(z_2) \le 0, \ldots$, $vD(z_n) \le 0$, respectively. If one of this cone is empty P_5' is unbounded (by definition P_5' has a solution) and P_4 is also unbounded. Then by agreeing to assign the value of $-\infty$ to the maximum of dual P_5', if P_5' has no solution, and using the duality theorem we can rewrite problem P_5' as:

$$\text{Max}\{v \cdot (b - E \cdot z)/v \cdot D(z) \le c\} \quad z \in R^Z, \quad \text{where } Z = \{z_1, z_2, \ldots, z_n\}$$

This maximum is obtained at the vertex of each polytope $V(z) = \{v/vD(z) \le c\}$. Assuming that $V(z)$ is not empty for all z, we denote by $(v_1^{z_1}, \ldots, v_I^{z_1}), \ldots, (v_1^{z_n}, \ldots, v_T^{z_n})$ the vertices of the polytope $V(z)$ where $z \in \{z_1, z_2, \ldots, z_n\}$ then P_5' can be written:

$$\begin{gathered} \underset{i=1,\ldots,I}{\text{Max}}\{v_i \cdot (b - E \cdot z_1)\} \\ \cdots\cdot\cdot \\ \underset{t=1,\ldots,T}{\text{Max}}\{v_l \cdot (b - E \cdot z_n)\} \end{gathered}$$

These previous maximization problems turn out to be equivalent to the following linear program:

$$
\begin{aligned}
&\text{Min } F_4 = \xi \\
&\text{st.} \\
&v_1^{z_1}(b - E \cdot z_1) \le \xi \\
&\dots \\
&v_I^{z_1}(b - E \cdot z_1) \le \xi \\
&v_1^{z_2}(b - E \cdot z_2) \le \xi \\
&\dots \\
&v_J^{z_2}(b - E \cdot z_2) \le \xi \\
&\dots \\
&v_1^{z_n}(b - E \cdot z_n) \le \xi \\
&\dots \\
&v_T^{z_n}(b - E \cdot z_n) \le \xi \\
&\xi \in [-\infty, +\infty]
\end{aligned}
$$

Considering the inequalities (I) that ensure that P_5 does not have an empty solution set, the following formulation is obtained:

$$
\begin{aligned}
&\text{Min } F_4(\xi) = \xi \\
&\text{st.} \\
&v_1^{z_1}(b - E \cdot z_1) \le \xi \\
&\dots\dots.. \\
&v_I^{z_1}(b - E \cdot z_1) \le \xi \\
&\dots\dots.. \\
&v_1^{z_n}(b - E \cdot z_n) \le \xi \\
&\dots\dots.. \\
&v_T^{z_n}(b - E \cdot z_n) \le \xi \\
&u_1^{z_1}(b - E \cdot z_1) \le 0 \\
&\dots\dots.. \\
&u_P^{z_1}(b - E \cdot z_1) \le 0 \\
&\dots\dots.. \\
&u_1^{z_n}(b - E \cdot z_n) \le 0 \\
&\dots\dots.. \\
&u_R^{z_n}(b - E \cdot z_n) \le 0 \\
&\xi \in [-\infty, +\infty]
\end{aligned}
$$

Which is equivalent to the following problem (P_6) named master problem (MP):

$$MP : P_6 \begin{cases} \text{Min } F_4(\xi) = \xi \\ \text{st.} \\ v_1^{z1}(b - E \cdot z) \leq \xi \\ \ldots\ldots.. \\ v_I^{z1}(b - E \cdot z) \leq \xi \\ \ldots\ldots.. \\ v_1^{zn}(b - E \cdot z) \leq \xi \\ \ldots\ldots.. \\ v_T^{zn}(b - E \cdot z) \leq \xi \\ u_1^{z1}(b - E \cdot z) \leq 0 \\ \ldots\ldots.. \\ u_P^{z1}(b - E \cdot z) \leq 0 \\ \ldots\ldots.. \\ u_1^{zn}(b - E \cdot z) \leq 0 \\ \ldots\ldots.. \\ u_R^{zn}(b - E \cdot z) \leq 0 \\ z \in \{0, 1\}, \xi \in [-\infty, +\infty] \end{cases}$$

At each stage of the algorithm, only some constraints of P_6 are known explicitly which gives rise to a problem named Restricted Master Problem (RMP) and involves a subset of the constraints of P_6 (Master Problem). Let $(\overline{z}, \overline{\xi})$ be an optimal solution of RMP, $\overline{\xi}$ is a lower bound of optimal ξ^* such that $\overline{\xi} \leq \xi^*$. An upper bound can be taken by the resolution of P_4 or P_2 (which is a restriction of the initial MIBLP). The upper bound (UB) is updated when a lower optimal solution is obtained for the current P_4 (or P_2) compared to the current UB.

In Benders decomposition a necessary and sufficient condition for $(\overline{z}, \overline{\xi})$ to be an optimal solution of P_6 is that $(\overline{z}, \overline{\xi})$ satisfy all the constraints of P_6 which are not explicitly stated in RMP but this condition cannot be satisfied in this approach. Another sufficient condition for the $(\overline{z}, \overline{\xi})$ to be an optimal solution of P_6 is that the $(UB - LB < \varepsilon)$ because the RMP is a relaxation of the original problem whereas the SP represent a restriction. Except for the case where RMP or the dual of P_5' do not have a feasible solution, in each iteration of the algorithm three cases arise:

4.1 Case 1 (Production of feasibility cut)

The optimal value of dual P_5 is unbounded. The Simplex algorithm is applied to dual P_5'' and produces an extreme ray u such that $\overline{u} \cdot (b - E \cdot \overline{z}) > 0$ and $\overline{u} \cdot D \leq 0$. Thus the constraint $\overline{u} \cdot (b - E \cdot z) \leq 0$ does not hold for the current solution $\overline{z}$ of RMP $(\overline{z}, \overline{\xi})$ and thus it is not a solution of P_6. The constraints $\overline{u} \cdot (b - E \cdot z) \leq 0$ must be added to RMP to form a new RMP augmented.

4.2 Case 2 (Production of integer exclusion cut)

The optimal value of dual has a finite value and $\overline{v} \cdot (b - E \cdot \overline{z}) - \xi \leq 0$. Since this constraint is satisfied, adding this constraint to RMP does not change the optimal value. If the optimality condition $(UB - LB < \varepsilon)$ is not satisfied the algorithm continues by excluding the current integer solution using the following cut:

$$\sum_{i \in P} z_i - \sum_{j \in Q} z_j \leq |P| - 1 \quad \text{(integer exclusion cut)}$$

where P is the set of indices of variables that have assumed the value 1, $P = \left\{i/z_i^* = 1\right\}$. Similarly Q is the set of indices where the corresponding variable assumed the value 0, $Q = \left\{j/z_j^* = 0\right\}$. By $|P|$ we denote the number of variables z_i^* that are equal to one. Adding this constraint to RMP the algorithm proceeds by generating a new integer optimal solution which gives rise to new SP.

4.3 Case 3 (Production of optimality cut)

The optimum of dual P_5 is bounded but contrary to case 2, we have $\overline{v} \cdot (b - E \cdot \overline{z}) - \xi > 0$. This shows that the constraint $\overline{v} \cdot (b - E \cdot \overline{z}) - \xi \leq 0$ is not satisfied by the current solution $(\overline{z}, \overline{\xi})$ of RMP and the constraint $\overline{v} \cdot (b - E \cdot z) - \xi \leq 0$ must therefore be added to P_6 in order to form a new augmented RMP.

In each iteration of the algorithm a *LB* is provided from the RMP and an UB is provided from the SP. The convergence criterion of the algorithm is when the difference between UB and LB is lower or equal to the parameter ε (UB $-$ LB $< \varepsilon$) where ε is a very small number.

We notice that in the case where we have constraints in the upper level optimization problem the proposed algorithm finds the optimal solution without any difficulty. The upper level constraints of the integer decision variables are added to the RMP and the upper level constraints of the integer and continuous decision variables are added to the SP. For the case that the upper level constraints involving both integer and continuous decision variables the extended KKT approach [24] is used in order to find the global optimal of the bi-level linear problem (SP). In case where the integer decisions variables appear into the upper level objective function as $c_3 z$ term then the variable ξ in the produced optimality cuts is replaced by $\xi - c_3 z$ ($\xi = \xi - c_3 z$).

To conclude the presentation of the algorithm we discuss the differences between the classical Benber's decomposition method for the mixed integer linear programs and the algorithm proposed in this paper. In the classical algorithm the dual of each of the slave problem (SP) has the same solution space. The difference between the slave problems is only in the objective function and the algorithm searches in each iteration the optimal dual solution in the same solution space going form one extreme point to another. In the proposed algorithm the difference between the slave problems is not only in the objective function but also in the range of dual variables. This difference influences the convergence of the algorithm. In classical Benders algorithm the optimal solution is determined when the solution of RMP results in $\overline{v} \cdot (b - E \cdot \overline{z}) - \xi \leq 0$ (case: 2).

In the presented algorithm it is not certain that in this case all the inequalities of P_6 are satisfied. The only group of inequalities satisfied is the ones that correspond to the current value of $z = \overline{z}$. Since a valid cut cannot be produced in this case following the classic Benders decomposition, the algorithm continues by excluding the current integer solution including an additional constraint (integer exclusion cut). Thus the convergence criterion for the proposed algorithm has to be modified to the difference between upper and lower bounds to be less than a small tolerance value ε.

Notice that the finite convergence of the algorithm results from the fact that problem P_6 has a finite number of constraints. The finite number of cuts results from the finite number of extreme points and rays of the dual P_5' and P_5'', respectively.

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